Pulling Two Correlated Normally-Distributed Random Variates
Part III - The Bivariate Normal Distribution

Gary Schurman, MBE, CFA

July, 2011

There are many instances especially with Monte Carlo simulations where we want to pull random variates from normal distributions that are correlated. We will work through the process and develop the attendant mathematics applicable to pulling two correlated random variates from the Bivariate Normal Distribution.

Our Hypothetical Problem

Assume that we want to pull the random variate x from a normal distribution with the following mean (μ_x) and variance (σ_x^2)...

$$x \sim N\left[\mu_x, \sigma_x^2\right] \sim N\left[3, 16\right] \tag{1}$$

Assume also that we want to pull the random variate y from a normal distribution with the following mean (μ_y) and variance (σ_y^2)...

$$y \sim N\left[\mu_y, \sigma_y^2\right] \sim N\left[5, 36\right] \tag{2}$$

Assume also that the correlation (ρ_{xy}) between the distributions of x and y is...

$$\rho_{x,y} = 0.80 \tag{3}$$

Question: We have a Monte Carlo simulation where we need to pull the random variate x from the distribution defined by Equation (1) above and the random variate y from the distribution defined by Equation (2) above noting that these two distributions are correlated as per Equation (3) above. Given that we have already pulled the independent random variates 0.60 and -1.25 from a normal distribution with mean zero and variance one what are the values of x and y that we will use in our Monte Carlo simulation?

Normalizing The Random Variates

We will first define the independent, normally-distributed random variate z_1, which has the following mean (μ_1) and variance (σ_1^2)...

$$z_1 \sim N\left[\mu_1, \sigma_1^2\right] \sim N\left[0, 1\right] \tag{4}$$

Using Equations (1) and (4) above we normalize the random variate x such that the equation for x becomes...

$$\begin{aligned} x &= \mu_x + x - \mu_x \\ &= \mu_x + \sigma_x \, \frac{x - \mu_x}{\sigma_x} \\ &= \mu_x + \sigma_x \, z_1 \end{aligned} \tag{5}$$

We now define the independent, normally-distributed random variate z_2, which has the following mean (μ_2) and variance (σ_2^2)...

$$z_2 \sim N\left[\mu_2, \sigma_2^2\right] \sim N\left[0, 1\right] \tag{6}$$

Using Equations (2) and (6) above we normalize the random variate y such that the equation for y becomes...

$$\begin{aligned} y &= \mu_y + y - \mu_y \\ &= \mu_y + \sigma_y \, \frac{y - \mu_y}{\sigma_y} \\ &= \mu_y + \sigma_y \, z_2 \end{aligned} \tag{7}$$

Note that normalizing the two normally-distributed random variates x and y does not affect correlation such that the correlation between x and y equals the correlation between z_1 and z_2. This statement in equation form is...

$$\rho_{x,y} = \rho_{z_1,z_2} \tag{8}$$

Correlating The Normalized Random Variates

We will use Ordinary Least Squares (OLS) to redefine the random variate z_2 from Equation (6) above such that it is dependent on the random variate z_1 from Equation (4) above. Using OLS we will redefine the normally-distributed random varate z_2 as...

$$z_2 = \alpha + \beta \, z_1 + \epsilon \tag{9}$$

The equation for the coefficient beta (β) in Equation (9) above using the standard OLS definition of beta and Equation (6) above is...

$$\beta = \frac{Cov(z_1, z_2)}{\sigma_2^2} = \frac{Cov(z_1, z_2)}{1} = Cov(z_1, z_2) \tag{10}$$

Using Equations (4) and (6) above the equation for the correlation between the random variates z_1 and z_2 is...

$$\rho_{z_1,z_2} = \frac{Cov(z_1, z_2)}{\sigma_1 \sigma_2} = \frac{Cov(z_1, z_2)}{1 \times 1} = Cov(z_1, z_2) \tag{11}$$

Using Equations (10) and (11) the coefficient beta (β) in Equation (9) can be defined as...

$$\beta = \rho_{z_1,z_2} \tag{12}$$

Using Equation (8) above we can rewrite Equation (12) as...

$$\beta = \rho_{x,y} \tag{13}$$

The equation for the constant alpha (α) in Equation (9) above using the standard OLS definition of alpha and Equations (4), (6) and (13) above is...

$$\begin{aligned} \alpha &= \mu_2 - \beta \, \mu_1 \\ &= 0 - (\rho_{x,y})(0) \\ &= 0 \end{aligned} \tag{14}$$

The mean (μ_ϵ) and variance (σ_ϵ^2) of the error term epsilon (ϵ) in Equation (9) above using the standard OLS definition of the error term is...

$$\epsilon \sim N\bigg[\mu_\epsilon, \sigma_\epsilon^2\bigg] \sim N\bigg[0, (1 - \rho_{z_1,z_2}^2) \, \sigma_2^2\bigg] \tag{15}$$

Using Equations (6) and (8) above we can rewrite Equation (15) as...

$$\epsilon \sim N\bigg[0, 1 - \rho_{x,y}^2\bigg] \tag{16}$$

We now define the independent, normally-distributed random variate z_3, which has the following mean (μ_3) and variance (σ_3^2)...

$$z_3 \sim N\bigg[\mu_3, \sigma_3^2\bigg] \sim N\bigg[0, 1\bigg] \tag{17}$$

Using Equations (16) and (17) above we will normalize the error term ϵ such that the equation for ϵ becomes...

$$\begin{aligned}\epsilon &= \mu_\epsilon + \epsilon - \mu_\epsilon \\ &= \mu_\epsilon + \sigma_\epsilon \, \frac{\epsilon - \mu_\epsilon}{\sigma_\epsilon} \\ &= \mu_\epsilon + \sqrt{1 - \rho^2_{x,y}} \times \frac{\epsilon - \mu_\epsilon}{\sqrt{1 - \rho^2_{x,y}}} \\ &= z_3 \, \sqrt{1 - \rho^2_{x,y}} \end{aligned} \tag{18}$$

Using Equations (13), (14) and (18) we can rewrite Equation (9) as...

$$\begin{aligned} z_2 &= \alpha + \beta \, z_1 + \epsilon \\ &= 0 + z_1 \, \rho_{x,y} + z_3 \sqrt{1 - \rho^2_{x,y}} \\ &= z_1 \, \rho_{x,y} + z_3 \, \sqrt{1 - \rho^2_{x,y}} \end{aligned} \tag{19}$$

The Solution To Our Hypothetical Problem

As outlined in the hypothetical problem above we pulled two independent, normally-distributed random variates with values 0.60 and -1.25 from a normal distribution with mean zero and variance one. Since the random variates z_1, as defined by Equation (4), and z_3, as defined by Equation (17), have this same distribution then we will define the random variates z_1 and z_3 as...

$$z_1 = 0.60 \;\; \text{...and...} \;\; z_3 = -1.25 \tag{20}$$

Using Equations (3) and (20) above then per Equation (19) the random variate z_2 is equal to...

$$\begin{aligned} z_2 &= z_1 \, \rho_{x,y} + z_3 \sqrt{1 - \rho^2_{x,y}} \\ &= (0.60)(0.80) + (-1.25)(\sqrt{1 - 0.80^2}) \\ &= 0.48 - 0.75 \\ &= -0.27 \end{aligned} \tag{21}$$

Using Equations (1) and (20) above then per Equation (5) the random variate x is equal to...

$$\begin{aligned} x &= \mu_x + \sigma_x \, z_1 \\ &= 3 + (\sqrt{16})(0.60) \\ &= 5.40 \end{aligned} \tag{22}$$

Using Equations (2) and (21) above then per Equation (7) the random variate y is equal to...

$$\begin{aligned} y &= \mu_y + \sigma_y \, z_2 \\ &= 5 + (\sqrt{36})(-0.27) \\ &= 3.38 \end{aligned} \tag{23}$$

The Solution: The solution to our problem is that the normally-distributed random variate $x = 5.40$ and the normally-distributed random variate $y = 3.38$.